

CLASSIFICATION OF COMPLEX CYCLIC LEIBNIZ ALGEBRAS

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ABSTRACT. Since Leibniz algebras were introduced by Loday as a generalization of Lie algebras, there has been a lot of interest in which results of the latter extend to the former. Cyclic Leibniz algebras, those generated by one element, are a useful tool for this purpose. In fact, they have no Lie algebra counterpart. Their simple structure lends itself to elegant counterexamples to the extension of several important results from Lie algebras to Leibniz algebras. In this paper, we give a classification of complex cyclic Leibniz algebras.

1. INTRODUCTION

Cyclic Leibniz algebras were introduced in [6] and appear in the classification of elementary [3] and minimal non-elementary Leibniz algebras [7]. They are also used as examples in the expository article [4]. In this work we classify these algebras in the complex case. Good references are [1], [2], [4], and [5].

We recall that a Leibniz algebra is an algebra in which left multiplication by every element is a derivation, i.e., multiplication satisfies $x(yz) = (xy)z + y(xz)$ for all $x, y, z \in \mathbf{A}$. Note that with the further constraint $xy = -yx$ this becomes the definition of a Lie algebra.

One major difference between Leibniz and Lie algebras is that the product of an element with itself in a Leibniz algebra may not be zero. Thus it makes sense to speak of Leibniz algebras generated by a single element. Such algebras are called cyclic Leibniz algebras. Several interesting results about these algebras have already been obtained. For instance, cyclic Leibniz algebras have a unique Cartan subalgebra, they have only finitely many maximal subalgebras, and all of these subalgebras can be explicitly computed [6].

2. BASIC STRUCTURE

Let \mathbf{A} be an n -dimensional vector space over \mathbb{C} containing a nonzero element a . Choose a linear operator $T : \mathbf{A} \rightarrow \mathbf{A}$ such that a is a cyclic vector for T , i.e., such that $\mathcal{B} = \{a, T(a), \dots, T^{n-1}(a)\}$ is a basis for \mathbf{A} . Then $T^n(a) = \alpha_1 a + \alpha_2 T(a) + \dots + \alpha_n T^{n-1}(a)$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. We define a product $\mathbb{C}a \times \mathbf{A} \rightarrow \mathbf{A}$ as follows: $(ca)v = cT(v)$ for all $v \in \mathbf{A}$ and $c \in \mathbb{C}$, i.e., such that T is left multiplication by a . Throughout the rest of the paper we will adopt the notation L_a when referring to this T . To avoid always writing the basis elements of \mathbf{A} in terms of L_a , we let $a^k = L_a^{k-1}(a)$. We aim to extend this product linearly to all of $\mathbf{A} \times \mathbf{A}$ in such a way that left multiplication is a derivation, or in other words, such that \mathbf{A} is a Leibniz algebra.

Proposition 2.1. *In the setting defined above, left multiplication is a derivation (\mathbf{A} is a Leibniz algebra) if and only if $L_{a^2} = 0$.*

Proof. Assume that left multiplication is a derivation. Then it is easy to check that $L_{a^2} = [L_a, L_a] = 0$, where $[\cdot, \cdot]$ is the commutator bracket of the Lie algebra $\text{Der}(\mathbf{A})$.

Now assume $L_{a^2} = 0$. By definition $L_a^j(x) = L_a(L_a^{j-1}(x))$ for all $j \geq 2$ and for all $x \in \mathbf{A}$. So by induction $L_a^j = 0$ for all $j \geq 2$. Now let $x = c_1 a + c_2 a^2 + \dots + c_n a^n \in \mathbf{A}$. Then by linearity and the fact that $L_{a^j} = 0$ for all $j \geq 2$, we have $L_x = c_1 L_a$. Thus it is enough to show that L_a is a derivation on \mathbf{A} . Since \mathbf{A} has basis $\{a, a^2, \dots, a^n\}$, we need only check that

$$(1) \quad L_a(a^i a^j) = L_a(a^i) a^j + a^i L_a(a^j) \quad \text{for all } 1 \leq i, j \leq n.$$

Both sides of (1) are zero when $i > 1$ and a^{j+2} when $i = 1$. Thus L_a is a derivation. \square

We will make use of the following consequence of Proposition 2.1: if \mathbf{A} is Leibniz, then $0 = a^{n+1}a = (\alpha_1 a + \alpha_2 a^2 + \dots + \alpha_n a^n)a = \alpha_1 a^2$. Thus $\alpha_1 = 0$.

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Of course, all of what we have said so far is well known. We could have just started by saying that a cyclic Leibniz algebra is an algebra generated by a single element a such that left multiplication by a is a derivation. We chose to include the preceding details because the underlying vector space structure is the heart of the following classification.

3. CLASSIFICATION OF CYCLIC LEIBNIZ ALGEBRAS OVER \mathbb{C}

Let \mathbf{A} be an n -dimensional cyclic Leibniz algebra over \mathbb{C} generated by a single element a . If $aa^n = 0$, then \mathbf{A} is the nilpotent cyclic Leibniz algebra of dimension n of which there is only one up to isomorphism. Throughout the rest of this paper, we consider only non-nilpotent cyclic Leibniz algebras, i.e., cyclic Leibniz algebras where the generator a satisfies

$$(2) \quad aa^n = \alpha_k a^k + \alpha_{k+1} a^{k+1} + \cdots + \alpha_n a^n$$

for some $2 \leq k \leq n$ and $\alpha_k \neq 0$.

From the discussion in the previous section, it is clear that any choice of $\alpha_2, \dots, \alpha_n$ defines a cyclic Leibniz algebra. However, differing choices of these coefficients do not always yield non-isomorphic algebras. A simple example is when \mathbf{A} is 2-dimensional and $aa^2 = \alpha a^2$ with $\alpha \neq 0, 1$. Let $x = \frac{1}{\alpha}a$. Then clearly x is a cyclic generator for \mathbf{A} and

$$(3) \quad xx^2 = \frac{1}{\alpha^3}aa^2 = \frac{1}{\alpha^2}a^2 = x^2 \neq \alpha x^2.$$

Thus \mathbf{A} itself has generators whose multiplications are different.

We consider the question of when two cyclic Leibniz algebras of the same dimension are isomorphic.

Lemma 3.1. *Let \mathbf{A} and \mathbf{B} be two cyclic Leibniz algebras of dimension n . Assume \mathbf{A} has a cyclic generator a which satisfies $aa^n = \alpha_k a^k + \alpha_{k+1} a^{k+1} + \cdots + \alpha_n a^n$. Then \mathbf{A} is isomorphic to \mathbf{B} if and only if \mathbf{B} has a cyclic generator b which satisfies $bb^n = \alpha_k b^k + \alpha_{k+1} b^{k+1} + \cdots + \alpha_n b^n$.*

Proof. Suppose there is an isomorphism of algebras $f : \mathbf{A} \rightarrow \mathbf{B}$. Then $b = f(a)$ satisfies $bb^n = \alpha_k b^k + \alpha_{k+1} b^{k+1} + \cdots + \alpha_n b^n$. Clearly b generates \mathbf{B} , since f is also a vector space isomorphism.

For the other implication, suppose \mathbf{B} has a generator b with the above multiplication. Let $f : \mathbf{A} \rightarrow \mathbf{B}$ be the vector space isomorphism that sends $a^i \mapsto b^i$ for $1 \leq i \leq n$. To show that this is a homomorphism of Leibniz algebras, we find that $f(aa^i) = f(a^{i+1}) = b^{i+1} = bb^i = f(a)f(a^i)$ for $1 \leq i < n$ and $f(aa^n) = f(\alpha_k a^k + \alpha_{k+1} a^{k+1} + \cdots + \alpha_n a^n) = \alpha_k f(a^k) + \alpha_{k+1} f(a^{k+1}) + \cdots + \alpha_n f(a^n) = \alpha_k b^k + \alpha_{k+1} b^{k+1} + \cdots + \alpha_n b^n = bb^n = f(a)f(a^n)$. From this we see that f respects all non-zero products in \mathbf{A} . Thus f is an isomorphism of Leibniz algebras. \square

Given a cyclic Leibniz algebra \mathbf{A} of dimension n with a generator a satisfying (2), we aim to find the isomorphism class of \mathbf{A} . By Lemma 3.1, it is enough to find all possible coefficients $\gamma_2, \dots, \gamma_n \in \mathbb{C}$ such that there exists a generator $x \in \mathbf{A}$ satisfying $xx^n = \gamma_2 x^2 + \gamma_3 x^3 + \cdots + \gamma_n x^n$.

Since a is a cyclic vector for L_a , it follows that L_a has characteristic polynomial $f(t) = t^n - \alpha_n t^{n-1} - \alpha_{n-1} t^{n-2} - \cdots - \alpha_k t^{k-1}$. By the Cayley-Hamilton theorem $f(L_a) = 0$. In other words

$$(4) \quad (L_a^n - \alpha_n L_a^{n-1} - \alpha_{n-1} L_a^{n-2} - \cdots - \alpha_k L_a^{k-1})(x) = 0$$

for all $x \in \mathbf{A}$. Now let us assume x is a cyclic generator and write x in terms of the basis \mathcal{B} : $x = c_1 a + c_2 a^2 + \cdots + c_n a^n$. By rearranging (4) we obtain

$$L_a^n(x) = \alpha_k L_a^{k-1}(x) + \alpha_{k+1} L_a^k(x) + \cdots + \alpha_n L_a^{n-1}(x).$$

Note that $c_1 \neq 0$ else x is not a cyclic generator for \mathbf{A} . We multiply by c_1^n which gives

$$c_1^n L_a^n(x) = c_1^n \alpha_k L_a^{k-1}(x) + c_1^n \alpha_{k+1} L_a^k(x) + \cdots + c_1^n \alpha_n L_a^{n-1}(x).$$

From the proof of Proposition 2.1, we know that $L_x = c_1 L_a$. Then

$$L_x^n(x) = c_1^{n-k+1} \alpha_k L_x^{k-1}(x) + c_1^{n-k} \alpha_{k+1} L_x^k(x) + \cdots + c_1 \alpha_n L_x^{n-1}(x),$$

which we may also write as

$$(5) \quad xx^n = c_1^{n-k+1} \alpha_k x^k + c_1^{n-k} \alpha_{k+1} x^{k+1} + \cdots + c_1 \alpha_n x^n.$$

Thus every generator for \mathbf{A} satisfies (5). Since $x = c_1 a$ is a generator for all $c_1 \neq 0$, there is at least one cyclic generator for \mathbf{A} satisfying (5) for every $c_1 \neq 0$. We have proven the following lemma:

Lemma 3.2. *Let \mathbf{A} be an n -dimensional cyclic Leibniz algebra with generator a satisfying (2). Then*

(i) *Every generator $x = c_1 a + \cdots + c_n a^n$ for \mathbf{A} satisfies*

$$(6) \quad xx^n = c_1^{n-k+1} \alpha_k x^k + c_1^{n-k} \alpha_{k+1} x^{k+1} + \cdots + c_1 \alpha_n x^n.$$

(ii) *\mathbf{A} has at least one generator satisfying (6) for each $c_1 \neq 0$.*

We call an n -dimensional non-nilpotent cyclic Leibniz algebra \mathbf{A} a *type k cyclic Leibniz algebra* if \mathbf{A} has a generator x with multiplication

$$(7) \quad xx^n = x^k + \gamma_{k+1} x^{k+1} + \cdots + \gamma_n x^n$$

for some ordered $(n-k)$ -tuple $(\gamma_{k+1}, \dots, \gamma_n) \in \mathbb{C}^{n-k}$.

Lemma 3.3. *Every n -dimensional non-nilpotent cyclic Leibniz algebra is of type k for one and only one $k \in \{2, \dots, n\}$.*

Proof. Let \mathbf{A} be a non-nilpotent cyclic Leibniz algebra having a generator a with multiplication as in (2). Let $x = c_1 a$ where $c_1 = \alpha_k^{\frac{1}{k-n-1}}$. Then by Lemma 3.2 part (i), x is a generator satisfying (7) with $\gamma_{k+i} = c_1^{n-k+1-i} \alpha_{k+i}$. Thus \mathbf{A} is of type k for at least one k . That this k is unique again follows immediately from part (i) of Lemma 3.2, since $c_1 \neq 0$ and $\alpha_k \neq 0$ imply that $c_1^{n-k+1} \alpha_k \neq 0$. \square

By Lemma 3.3, we know that any non-nilpotent cyclic Leibniz algebra of type k has a generator satisfying (7) for some $(n-k)$ -tuple $(\gamma_{k+1}, \dots, \gamma_n)$. The question remains as to whether \mathbf{A} can also have a generator satisfying (7) for some other $(n-k)$ -tuple $(\gamma'_{k+1}, \dots, \gamma'_n)$.

Let $d = n - k$. We define the following relation on \mathbb{C}^d . We say $(\gamma_1, \gamma_2, \dots, \gamma_d) \sim (\gamma'_1, \gamma'_2, \dots, \gamma'_d)$ if $(\gamma_1, \gamma_2, \dots, \gamma_d) = (\omega^d \gamma'_1, \omega^{d-1} \gamma'_2, \dots, \omega \gamma'_d)$ for some $(d+1)$ -th root of unity ω . One may easily check that \sim is an equivalence relation on \mathbb{C}^d . Then the equivalence classes are of the form

$$[(\gamma_1, \gamma_2, \dots, \gamma_d)] = \{(\omega^d \gamma_1, \omega^{d-1} \gamma_2, \dots, \omega \gamma_d) \mid \omega \text{ is an } (d+1)\text{-th root of unity}\}.$$

Then we have the following lemma.

Lemma 3.4. *Fix $k \in \{2, \dots, n\}$ and let $(\gamma_{k+1}, \dots, \gamma_n) \in (\mathbb{C}^{n-k}) \setminus 0$. Let \mathbf{A} be a cyclic Leibniz algebra of dimension n containing a generator x such that $xx^n = x^k + \gamma_{k+1} x^{k+1} + \cdots + \gamma_n x^n$. Then \mathbf{A} also has a generator y such that $yy^n = y^k + \gamma'_{k+1} y^{k+1} + \cdots + \gamma'_n y^n$ if and only if $(\gamma'_{k+1}, \dots, \gamma'_n) \sim (\gamma_{k+1}, \dots, \gamma_n)$, where \sim is the equivalence relation on \mathbb{C}^{n-k} defined above.*

Proof. Assume y satisfies the equation given above and write $y = c_1 x + c_2 x^2 + \cdots + c_n x^n$. Then Lemma 3.2 says that $c_1^{n-k+1} = 1$. Then c_1 is an $(n-k+1)$ -th root of unity, and by Lemma 3.2 part (i) we have $yy^n = y^k + c_1^{n-k} \gamma_{k+1} y^{k+1} + \cdots + c_1 \gamma_n y^n$. Thus $(\gamma_{k+1}, \dots, \gamma_n) \sim (\gamma'_{k+1}, \dots, \gamma'_n)$.

For the other implication, let $(\gamma'_{k+1}, \dots, \gamma'_n) \in [(\gamma_{k+1}, \dots, \gamma_n)]$. Then $(\gamma'_{k+1}, \dots, \gamma'_n) = (\omega^{n-k} \gamma_{k+1}, \dots, \omega \gamma_n)$ for some $(n-k+1)$ -th root of unity ω . Then the generator $y = \omega x$ satisfies $yy^n = y^k + \omega^{n-k} \gamma_{k+1} y^{k+1} + \cdots + \omega \gamma_n y^n = y^k + \gamma'_{k+1} y^{k+1} + \cdots + \gamma'_n y^n$. \square

We have shown that there is a one-to-one correspondence between the isomorphism classes of non-nilpotent n -dimensional cyclic Leibniz algebras of type k and the nonzero elements of \mathbb{C}^{n-k} / \sim . More precisely, we have the following classification:

Theorem 3.5 (Classification). *Let \mathbf{A} be an n -dimensional cyclic Leibniz algebra over \mathbb{C} . Then \mathbf{A} is isomorphic to a Leibniz algebra spanned by $\{a, a^2, \dots, a^n\}$ with the product aa^n given by one and only one of the following:*

- (1) $aa^n = 0$ (nilpotent case).
- (2) $aa^n = a^n$.
- (3) $aa^n = a^k + \alpha_{k+1} a^{k+1} + \cdots + \alpha_n a^n$, $2 \leq k \leq n-1$, $(\alpha_{k+1}, \dots, \alpha_n) \in \mathbb{C}^{n-k} / \sim$.

Proof. That there is only one n -dimensional nilpotent cyclic Leibniz algebra up to isomorphism follows from Lemma 3.1. Now assume \mathbf{A} is a non-nilpotent n -dimensional cyclic Leibniz algebra. Then by Lemma 3.3, \mathbf{A} has a generator satisfying one and only one of (2) or (3). Now assume \mathbf{A} has a generator a satisfying $aa^n = a^k + \alpha_{k+1}a^{k+1} + \cdots + \alpha_na^n$. Then by Lemma 3.4, \mathbf{A} also has a generator b satisfying $aa^n = a^k + \alpha'_{k+1}a^{k+1} + \cdots + \alpha'_na^n$ if and only if $(\alpha'_{k+1}, \dots, \alpha'_n) \sim (\alpha_{k+1}, \dots, \alpha_n)$. \square

We think it worth nothing that for each $k = 2, \dots, n-1$ there is an $(n-k)$ -parameter family of isomorphism classes of cyclic Leibniz algebras of type k , the parameters being chosen from the uncountable set \mathbb{C}^{n-k}/\sim . Thus for $n \geq 3$ there are uncountably many isomorphism classes of cyclic Leibniz algebras of dimension n over \mathbb{C} .

4. 3 AND 4-DIMENSIONAL CLASSIFICATION

We use the 3 and 4-dimensional cases to illustrate Theorem 3.5:

Corollary 4.1 (3-dimensional Classification). *Let \mathbf{A} be a 3-dimensional cyclic Leibniz algebra over \mathbb{C} . Then \mathbf{A} is isomorphic to a Leibniz algebra spanned by $\{a, a^2, a^3\}$ with the product aa^3 given by one and only one of the following:*

- (1) $aa^3 = 0$ (nilpotent case).
- (2) $aa^3 = a^3$.
- (3) $aa^3 = a^2 + \alpha_3a^3$, $\alpha_3 \in \mathbb{C}/\sim$, where $\alpha \sim \alpha'$ if $\alpha' = \pm\alpha$.

Corollary 4.2 (4-dimensional Classification). *Let \mathbf{A} be a 4-dimensional cyclic Leibniz algebra over \mathbb{C} . Then \mathbf{A} is isomorphic to a Leibniz algebra spanned by $\{a, a^2, a^3, a^4\}$ with the product aa^4 given by one and only one of the following:*

- (1) $aa^4 = 0$ (nilpotent case).
- (2) $aa^4 = a^4$.
- (3) $aa^4 = a^3 + \alpha_4a^4$, $\alpha_4 \in \mathbb{C}/\sim$, where $\alpha \sim \alpha'$ if $\alpha' = \pm\alpha$.
- (4) $aa^4 = a^2 + \alpha_3a^3 + \alpha_4a^4$, $(\alpha_3, \alpha_4) \in \mathbb{C}/\sim$, where $(\alpha, \beta) \sim (\alpha', \beta')$ if $(\alpha', \beta') \in \{(\alpha, \beta), (\omega^2\alpha, \omega\beta), (\omega\alpha, \omega^2\beta) \mid \omega = e^{2\pi i/3}\}$

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REFERENCES

- [1] Sh. A. Ayupov and B. A. Omirov. On Leibniz algebras. *Algebra and Operator Theory, Proceedings of the Colloquium in Tashkent, Kluwer, Dordrecht*, pages 1–13, 1998.
- [2] D.W. Barnes. Some theorems on Leibniz algebras. *Comm. Algebra*, 39(7):2463–2472, 2011.
- [3] C. Batten, L. Bosko-Dunbar, A. Hedges, J.T. Hird, K. Stagg, and E. Stitzinger. A Frattini theory for Leibniz algebras. *Comm. Algebra*, 41(4):1547–1557, 2013.
- [4] I. Demir, K.C. Misra, and E. Stitzinger. On some structures of Leibniz algebras. *Contemporary Mathematics*, 623:41–54, 2014.
- [5] V. Gorbatsevich. On some basic properties of Leibniz algebras. arXiv:1302.3345v2 [math.RA].
- [6] C.B. Ray, A. Combs, N. Gin, A. Hedges, J.T. Hird, and L. Zack. Nilpotent Lie and Leibniz algebras. *Comm. Algebra*, 42:2404–2410, 2014.
- [7] C.B. Ray, A. Hedges, and E. Stitzinger. Classifying several classes of Leibniz algebras. *Algebras and Representation Theory*, 17:703–712, 2014.

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